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# CYCLES OF NONZERO ELEMENTS IN LOW RANK MATRICES

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*Dedicated to the memory of Paul Erdős*

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We consider the problem of finding some structure in the zero-nonzero pattern of a low rank matrix. This problem has strong motivation from theoretical computer science. Firstly, the well-known problem on rigidity of matrices, proposed by Valiant as a means to prove lower bounds on some algebraic circuits, is of this type. Secondly, several problems in communication complexity are also of this type. The special case of this problem, where one considers positive semidefinite matrices, is equivalent to the question of arrangements of vectors in euclidean space so that some condition on orthogonality holds. The latter question has been considered by several authors in combinatorics [1,4]. Furthermore, we can think of this problem as a kind of Ramsey problem, where we study the tradeoff between the rank of the adjacency matrix and, say, the size of a largest complete subgraph. In this paper we show that for an $n \times n$ real matrix with nonzero elements on the main diagonal, if the rank is $o(n)$, the graph of the nonzero elements of the matrix contains certain cycles. We get more information for positive semidefinite matrices.

## 1. Introduction

This paper is a contribution to an important general problem to find structural properties of matrices caused by their low rank. The structure that we are interested in is given by the zero vs. nonzero pattern of the matrix. We shall consider only square matrices, thus the pattern can be described by a directed graph. Our motivation is from theoretical computer science. In

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particular the aim of this research is to prove lower bounds on the rigidity of matrices. The rigidity of a matrix was introduced as a means to prove lower bounds on the size of certain algebraic circuits by Valiant [6]. In [2] an explicit matrix has been constructed which has large rigidity provided one can prove certain properties of low rank matrices. The results in this paper go in this direction, but they are not strong enough for this purpose. For positive semidefinite matrices our problem can be equivalently stated as a problem of arrangements of vectors in a low dimensional euclidean space so that given pairs of vectors are orthogonal [4,1]. Another related well-known problem is the relation of the rank of a 0-1 matrix to its communication complexity (see [3] for some recent results).

In this paper we consider square matrices over reals (but most results can be probably generalized to complex numbers). By low rank we mean rank less than $\varepsilon n$, where the matrix is $n \times n$ and $\varepsilon > 0$ is a constant (less formally, we can state it as rank being $o(n)$). We would like to show that such a condition, for sufficiently small $\varepsilon$, implies a certain pattern of nonzero elements of the matrix. In general, low rank does not imply the existence of nonzero elements (consider the zero matrix). Therefore we have to assume that some nonzero elements are present and deduce the existence of others from it. Our assumption will be that all elements on the main diagonal are nonzero. We shall show that such low rank matrices contain rectangles whose vertices are nonzeros and, assuming that all elements are nonnegative, they contain rectangular hexagons whose vertices are nonzeros. In case of rectangles this simply means a $2 \times 2$ submatrix of nonzero elements; in case of hexagons it means a $3 \times 3$ submatrix with all nonzero elements except possibly for elements on some diagonal. Note that for these results we may use an assumption more general than the nonzeros on the main diagonal, namely, that the matrix has a support.

An alternative way of looking at the structure of nonzero elements is to define, for a square matrix $A = (a_{ij})$, a directed graph $G_A$ which determines the nonzero structure of the matrix. Thus $(i,j)$ is an edge in $G_A$ iff $a_{ij} \neq 0$. A rectangular $k$-gon of nonzero elements in $A$, where none of the elements is on the main diagonal, corresponds to a circuit in $G_A$ where the direction of edges alternate. If a vertex of such a $k$-gon happens to be at the main diagonal, then the corresponding edge is just a loop. If $A$ is a symmetric matrix, then we can view $G_A$ as undirected graph. Then, in particular, a rectangle with one vertex on the main diagonal corresponds to a triangle in $G_A$ and a rectangular hexagon with one vertex on the main diagonal corresponds to a 5-cycle. We shall show that such configurations occur in matrices of low rank which are positive semidefinite.

Because of possible applications to the rigidity of matrices we are primarily interested in matrices whose entries are arbitrary reals, but the results have consequences also for the problem of relating the spectrum of a graph to its properties. E.g. Theorem 6 implies that a nonempty graph whose least eigenvalue has multiplicity bigger than 1/2 of the number of vertices must contain a triangle.

The results for rectangles have been proved in [2] respectively [4]. We present them here to show that all these results can be proved by the same method. The main result is the theorem about 5-cycles in positive semidefinite matrices (Theorem 10). Our method is based on inequalities for the rank of a matrix in terms of the traces of powers of the matrix, which are proved using eigenvalues. The idea of using eigenvalues for this type of problems is due to Rosenfeld [4].

## 2. Inequalities and other auxiliary results

The following is a generalization of the often used inequality

$$\operatorname{rank} A \geq (\operatorname{tr} A)^2/\operatorname{tr}(A^2). \tag{1}$$

**Lemma 1.** *Let $s>t>0$ be integers and $A$ be a symmetric matrix. Suppose that either $A$ is positive semidefinite, or $s/t$ is even, or $t$ is even. Then*

$$(\operatorname{rank} A)^{s-t} \geq \frac{(\operatorname{tr} A^t)^s}{(\operatorname{tr} A^s)^t}, \tag{2}$$

*provided that* $\operatorname{tr} A^s \neq 0$.

**Proof.** Let $\lambda_i$, for $i=1,\dots,n$, be the eigenvalues of $A$, $\lambda_1,\dots,\lambda_r \neq 0$, and $\lambda_{r+1},\dots,\lambda_n=0$, where $r$ is the rank of $A$. If $A$ is positive semidefinite, then the eigenvalues are nonnegative. By the convexity of the function $x^{\frac{s}{t}}$, we have for positive $x_1,\dots,x_r$,

$$r\left(\frac{\sum_{k=1}^r x_k}{r}\right)^{\frac{s}{t}} \leq \sum_{k=1}^r x_k^{\frac{s}{t}}.$$

Applying it to the $t$-th powers of the eigenvalues, we get

$$r\left(\frac{\sum_{k=1}^r \lambda_k^t}{r}\right)^{\frac{s}{t}} \leq \sum_{k=1}^r \left(\lambda_k^t\right)^{\frac{s}{t}},$$

whence

$$r^{s-t} \geq \frac{\left(\sum_k \lambda_k^t\right)^s}{\left(\sum_k \lambda_k^s\right)^t}.$$

The eigenvalues of $A^t$ and $A^s$ are the $t$-th resp. $s$-th powers of the eigenvalues of $A$, the trace is the sum of the eigenvalues, thus the lemma follows. ∎

Let us note that further generalizations are possible, eg., $s$ and $t$ can be arbitrary reals, if $A$ is positive semidefinite. We give another nice formula, due to Codenotti, though we do not have applications for it.

**Corollary 2.** *Let $i$, $j$ be integers and suppose that either $A$ is positive semidefinite, or $i$ and $j$ are even. Then*

$$\mathrm{rank} A \geq \frac{\mathrm{tr}(A^i)\mathrm{tr}(A^j)}{\mathrm{tr}(A^{i+j})}.$$

**Proof.** We consider two instances of (2), corresponding to $s=i+j$, $t=i$, and to $s=i+j$ and $t=j$. Multiplying term by term the two inequalities, we obtain

$$\mathrm{rank}(A)^j \mathrm{rank}(A)^i \geq \frac{(\mathrm{tr} A^i)^{i+j}}{(\mathrm{tr} A^{i+j})^i} \frac{(\mathrm{tr} A^j)^{i+j}}{(\mathrm{tr} A^{i+j})^j}.$$

Simplifying and taking the $(i+j)$-root we obtain the thesis. ∎

In the rest of this section we shall recall some well-known results that we shall need.

Schur's theorem states that the componentwise product of two positive semidefinite matrices is positive semidefinite. Given a matrix $B$, we denote by $B^{(2)}$ the componentwise product of $B$ with itself, i.e., the matrix $(b_{ij}^2)_{i,j}$.

By a theorem of Sinkhorn and Knopp [5], if $A$ if a matrix with nonnegative entries and $A$ is fully indecomposable (this means that $A$ does not contain a $k \times l$ submatrix of zeros with $k+l \geq n$, where $n$ is the dimension of $A$), then there exist diagonal matrices $D, E$ with positive entries on the main diagonals such that $DAE$ is doubly stochastic. This means that we can normalize the rows and columns of such matrices while preserving the rank and the structure of nonzero elements. Furthermore, the matrices $D$ and $E$ are uniquely determined.

The uniqueness clearly implies that $D=E$, if $A$ is moreover symmetric. Note that a symmetric matrix with nonzero elements on the main diagonal is a direct sum of fully indecomposable matrices. Hence, for symmetric matrices it suffices to assume that they have nonzero elements on the main diagonal in Sinkhorn-Knopp theorem. Thus we get the following lemma.

**Lemma 3.** *Let $A$ be a matrix with nonzero elements elements on the main diagonal.*

*(i) If $A$ is symmetric, then there is a diagonal matrix $D$ with positive entries on the main diagonal such that $(DAD)^{(2)}$ is doubly stochastic.*

*(ii) If $A$ is fully indecomposable, then there are diagonal matrices $D,E$ with positive entries on the main diagonals such that $(DAE)^{(2)}$ is doubly stochastic.*

**Proof.** (i) By the remarks above, there is a diagonal matrix $E$ with positive entries on the main diagonal such that $EA^{(2)}E$ is doubly stochastic. Put $D=\sqrt{E}$.

(ii) Apply the same trick with the square roots. ∎

Note that in (i), if $A$ is positive semidefinite, then also $DAD$ is positive semidefinite. (As we consider only real matrices, every positive semidefinite matrix is symmetric.)

In case of nonsymmetric matrices we shall use the following lemma, whose proof we leave to the reader.

**Lemma 4.** *Let $A$ be a matrix with nonzero elements on the main diagonal. Then the main diagonal can be covered by disjoint fully indecomposable matrices $A_i$ such that* $\operatorname{rank} A=\sum_i \operatorname{rank} A_i$.

## 3. Rectangles

**Theorem 5** ([2]). *Let $A$ be an $n\times n$ matrix with nonzero elements on the main diagonal. If* $\operatorname{rank} A<n/2$, *then $A$ contains a $2\times 2$ submatrix with nonzero elements.*

**Proof.** We include a short proof of this result, since it is a paradigm for the other results.

Suppose that $A$ does not contain a $2\times 2$ submatrix with nonzero elements. By Lemma 4 we can assume w.l.o.g. that $A$ is fully indecomposable. Multiplying a matrix by a full rank diagonal matrix does not change the pattern of zeros and nonzeros and the rank. Hence, by Lemma 3 (ii), we can assume that $A^{(2)}$ is already doubly stochastic. We shall apply the inequality (1) to $B=AA^\top$.

First we compute the square of the trace of $B$. Let $B=(b_{ij})_{ij}$, then $b_{ij}=\sum_k a_{ik}a_{jk}$. Thus

$$(\operatorname{tr} B)^2=\left(\sum_i b_{ii}\right)^2=\left(\sum_{i,k} a_{ik}^2\right)^2=n^2,$$

since $A^{(2)}$ is doubly stochastic. Now we estimate the trace of the square of $B$.

$$\mathrm{tr}B^2 = \mathrm{tr}BB^\top = \sum_{i,j} b_{ij}^2 = \sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2 = \sum_{i,j,k,l} a_{ik}a_{jk}a_{il}a_{jl}.$$

Since $A$ does not contain a $2\times 2$ submatrix with nonzero elements, the terms in the sum are nonzero only if $i=j$ or $k=l$. Thus the sum is equal to

$$\sum_{i,k,l} a_{ik}^2 a_{il}^2 + \sum_{i,j,k} a_{ik}^2 a_{jk}^2 - \sum_{i,k} a_{i,k}^4 \le \sum_{i,k,l} a_{ik}^2 a_{il}^2 + \sum_{i,j,k} a_{ik}^2 a_{jk}^2 =$$

$$\sum_i \left(\sum_k a_{ik}^2\right)^2 + \sum_k \left(\sum_i a_{ik}^2\right)^2 = 2n,$$

using the double stochasticity of the matrix $A^{(2)}$.
Thus $\mathrm{rank}A \ge (\mathrm{tr}B)^2/\mathrm{tr}B^2 \ge n/2$. ∎

The following is a result of Rosenfeld [4]. We give a slightly different proof, which is in line of our other results.

**Theorem 6.** *Let $A$ be an $n\times n$ positive semidefinite matrix with positive elements on the main diagonal. If $\mathrm{rank}A < n/2$, then $A$ contains a $2\times 2$ submatrix with nonzero elements and exactly one element on the main diagonal, i.e., $G_A$ contains a triangle.*

**Proof.** We shall prove the contrapositive implication. Suppose that $A$ does not contain a $2\times 2$ submatrix with nonzero elements and exactly one element on the main diagonal. Without loss of generality we also assume that $a_{ii}=1$ for $i=1,..,n$, because this condition can be achieved by multiplying $A$ by positive diagonal matrices. (Geometrically it means that we are considering an arrangement of unit vectors.)

Setting $s=3$ and $t=2$ in inequality (2) we obtain

$$\mathrm{rank}(A) \ge \frac{\left(\mathrm{tr}(A^2)\right)^3}{\left(\mathrm{tr}(A^3)\right)^2} = \frac{\left(\sum_{ij} a_{ij}a_{ji}\right)^3}{\left(\sum_{ijk} a_{ij}a_{jk}a_{ki}\right)^2},$$

The sum $\sum_{ij} a_{ij}a_{ji}$ can be decomposed as $\sum_i a_{ii}^2 + \sum_{i\ne j} a_{ij}a_{ji} = n + \sum_{i\ne j} a_{ij}^2$, since $a_{ii}=1$ and $A$ is symmetric.

Since $A$ does not contain a $2\times 2$ submatrix with nonzero elements and exactly one element on the main diagonal, at least one pair of indices in the sum $\sum_{ijk} a_{ij}a_{jk}a_{ki}$ must coincide. Thus the sum can be upper bounded by

$$\sum_i a_{ii}^3 + \sum_{i=j\neq k} a_{ii}a_{ik}a_{ki} + \sum_{k=i\neq j} a_{kj}a_{jk}a_{kk} + \sum_{j=k\neq i} a_{ij}a_{jj}a_{ji} = n + 3\sum_{i\neq j} a_{ij}^2\,,$$

hence

$$\operatorname{rank}(A) \geq \frac{(n+x)^3}{(n+3x)^2} \geq \frac{n}{2}\,,$$

where $x=\sum_{i\neq j} a_{ij}^2$. The last inequality follows from the fact that $(n+x)^3/(n+3x)^2-n/2\geq 0$ if and only if $(x-n)^2(2x+n)\geq 0$, which is always true, since $x$ is a sum of squares. ∎

Rosenfeld also gives an example showing that his bound is best possible. In [2] it has been shown that the assumption that $A$ is positive semidefinite is essential.

## 4. Hexagons

**Lemma 7.** *Let $A$ be an $n\times n$ matrix such that $A^{(2)}$ is stochastic. Then $\operatorname{tr}(AA^\top AA^\top)\geq n$.*

**Proof.**

$$\begin{aligned}\operatorname{tr}(AA^\top AA^\top) &= \sum_{ijkh} a_{ik}a_{jk}a_{jh}a_{ih} = \sum_{ij}\left(\sum_k a_{ik}a_{jk}\right)^2 \\ &\geq \sum_i\left(\sum_k a_{ik}^2\right)^2 = n\,.\end{aligned}$$

∎

**Theorem 8.** *Let $A$ be an $n\times n$ matrix with nonnegative entries and with nonzero elements on the main diagonal. If $\operatorname{rank} A<n/36$, then $A$ contains a rectangular hexagon whose vertices are nonzero elements.*

**Proof.** We use the following inequality, obtained from (2) by setting $s=3$ and $t=2$,

$$\operatorname{rank}(A) \geq \operatorname{rank}(B) \geq \frac{\left(\operatorname{tr}(B^2)\right)^3}{\left(\operatorname{tr}(B^3)\right)^2}\,,$$

where $B=AA^\top$.

W.l.o.g. we may assume that $A^{(2)}$ is doubly stochastic. We need to upper bound

$$\text{(3)} \qquad \mathrm{tr}(B^3) = \mathrm{tr}(AA^\top AA^\top AA^\top) = \sum_{i_1 \cdots i_6} a_{i_1 i_2} a_{i_3 i_2} a_{i_3 i_4} a_{i_5 i_4} a_{i_5 i_6} a_{i_1 i_6} \,.$$

If $A$ does not contain a hexagon all the terms corresponding to hexagons must be zero. The entries in the term $a_{i_1 i_2} a_{i_3 i_2} a_{i_3 i_4} a_{i_5 i_4} a_{i_5 i_6} a_{i_1 i_6}$ are contained in rows $i_1, i_3, i_5$ and columns $i_2, i_4, i_6$. The entries form a hexagon iff $i_1, i_3, i_5$ are distinct and $i_2, i_4, i_6$ are distinct. Thus our assumption means that the term is nonzero only if at least one of the six equalities $i_1 = i_3$, $i_1 = i_5$, $i_3 = i_5$, $i_2 = i_4$, $i_2 = i_6$, $i_4 = i_6$ holds.

Because of symmetry we need to estimate only the sum of terms where $i_1 = i_3$, which is:

$$\sum_{i_2 \cdots i_6} a_{i_3 i_2}^2 a_{i_3 i_4} a_{i_5 i_4} a_{i_5 i_6} a_{i_3 i_6} = \sum_{i_3 \cdots i_6} \left( \sum_{i_2} a_{i_3 i_2}^2 \right) a_{i_3 i_4} a_{i_5 i_4} a_{i_5 i_6} a_{i_3 i_6}$$

$$= \sum_{i_3 \cdots i_6} a_{i_3 i_4} a_{i_5 i_4} a_{i_5 i_6} a_{i_3 i_6} = \mathrm{tr}(B^2) \,.$$

To upper bound $\mathrm{tr}(B^3)$ we take 6 times this expression. In this way we over count some terms, as we take the terms where more pairs of indices are equal more times than they occur in $\mathrm{tr}(B^3)$. As we assume that all entries of $A$ are nonnegative, we are adding positive terms and thus we can upper bound the $\mathrm{tr}(B^3)$ in this way.

Hence, using Lemma 7, we have

$$\mathrm{rank}(A) \geq \mathrm{rank}(B) \geq \frac{(\mathrm{tr}(B^2))^3}{(6\mathrm{tr}(B^2))^2} \geq \frac{\mathrm{tr}(B^2)}{36} \geq \frac{n}{36} \,. \qquad \blacksquare$$

Now we consider 5-cycles in $G_A$. Again, the key step is to estimate a certain trace.

**Lemma 9.** *Let $A$ be an $n \times n$ positive semidefinite matrix such that $A^{(2)}$ is doubly stochastic. Suppose $G_A$ does not contain 5-cycles. Then*

$$\mathrm{tr}A^5 \leq c_1 \mathrm{tr}A^4 + c_2 \mathrm{tr}A^3 + c_3 n \,,$$

*where $c_1$, $c_2$, $c_3$ are positive constants.*

**Proof.** We consider $\mathrm{tr}A^5 = \sum_{i_0 i_1 i_2 i_3 i_4} a_{i_0 i_1} a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} a_{i_4 i_0}$, where all the terms for which all five indices pairwise different are zero. We would like to estimate it as above, i.e., by the sum of sums where each sum is over terms with particular two indices equal. As we do not assume that the entries are nonnegative, we have to be more careful, as in this way we may count some terms more then once. For instance, if three indices are equal, then we would count such a term three times. So we estimate this expression using the inclusion-exclusion formula. Namely, instead of dividing the terms into disjoint groups according to how many different indices occur in them, we write it as the sum of the following sums:

1. the sum of terms in which there are at most four different indices,
2. minus the sum of terms in which there are at most three different indices,
3. plus the sum of terms in which there are at most two different indices,
4. minus the sum of terms in which all indices are equal.

To prove the lemma, it suffices to upper bound the sums with a positive sign (cases 1 and 3), by constants times $\mathrm{tr}A^4, \mathrm{tr}A^3, n$ and to show that each of the sums which appears with the minus sign (cases 2 and 4) is nonnegative. The types of terms that appear after identification of certain number of indices can be identified with possible contractions of a pentagon. They are shown on Figure 1.

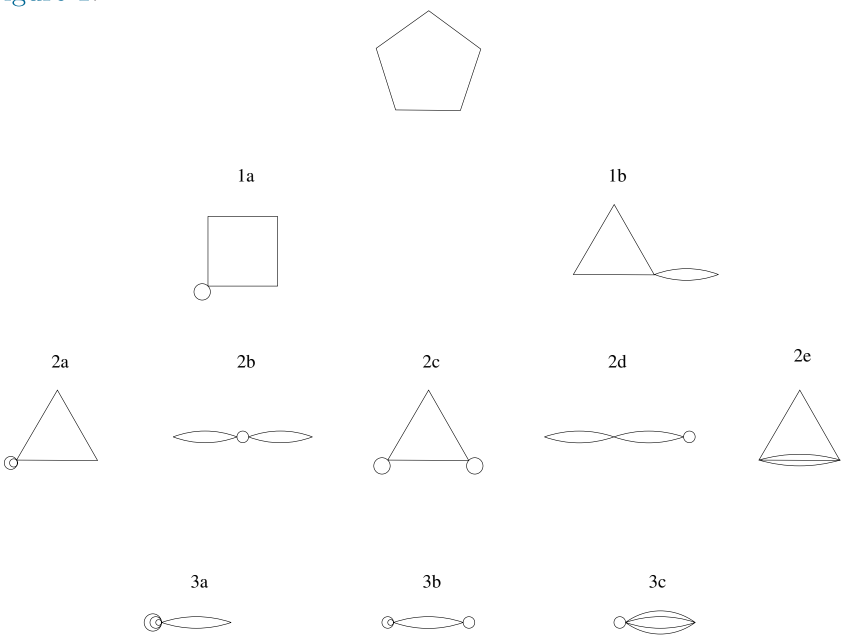


**Figure 1**

1. At most four indices.

(a) Two consecutive indices are equal. There are five isomorphic types of terms, one of them has the form $a_{i_0i_0}a_{i_0i_2}a_{i_2i_3}a_{i_3i_4}a_{i_4i_0}$, i.e., $i_0=i_1$. This is the situation, in which the pentagon degenerates to a 4-cycle with a loop; but as we do not exclude that more indices are the same, it may also be some configuration obtained from the 4-cycle and a loop by further contractions. For fixed $i_0$, consider the sum over $i_2,i_3,i_4$:

$$\sum_{i_2,i_3,i_4} a_{i_0i_0}a_{i_0i_2}a_{i_2i_3}a_{i_3i_4}a_{i_4i_0} \leq \sum_{i_2,i_3,i_4} a_{i_0i_2}a_{i_2i_3}a_{i_3i_4}a_{i_4i_0},$$

where we used $0\leq a_{i_0i_0}\leq 1$ (the first inequality follows from matrix $A$ being positive semidefinite, the second follows from double stochasticity of $A^{(2)}$). Note that this is a diagonal term in the matrix $A^4$. Hence summing the term over $i_0$, we obtain $\mathrm{tr}A^4$. Thus the sum of such terms is bounded by $5\mathrm{tr}A^4$.

(b) Two nonconsecutive indices are equal. In this case there are five types of terms; one of them is $a^2_{i_0i_1}a_{i_0i_3}a_{i_3i_4}a_{i_4i_0}$. This corresponds to a triangle with a double edge appended. Summing over $i_1$ with other indices fixed and using the double stochasticity, we get $a_{i_0i_3}a_{i_3i_4}a_{i_4i_0}$, which is a diagonal term of $A^3$. Hence the sum of such terms is bounded by $5\mathrm{tr}A^3$.

2. At most three different indices. (From now on we consider only one type of terms from each isomorphism group.)

(a) $i_0=i_1=i_2$, i.e., the terms are of the form $a^2_{i_0i_0}a_{i_0i_3}a_{i_3i_4}a_{i_4i_0}$. Consider the sum over $i_3$ and $i_4$ for a fixed $i_0$

$$\sum_{i_3,i_4} a^2_{i_0i_0}a_{i_0i_3}a_{i_3i_4}a_{i_4i_0} = a^2_{i_0i_0}\cdot (a_{i_0i_3})^\top_{i_3}\,(a_{i_3i_4})_{i_3i_4}\,(a_{i_0i_4})_{i_4}\,,$$

where we use parentheses to denote two vectors and a matrix. The above sum is nonnegative because the matrix $(a_{i_3i_4})_{i_3i_4}$ is actually the matrix $A$ which is positive semidefinite by assumption. Hence the sum of all such terms is nonnegative.

(b) $i_0=i_2=i_4$, i.e., the terms are of the form $a_{i_0i_0}a^2_{i_0i_1}a^2_{i_3i_0}$. These terms are clearly nonnegative, since the diagonal terms of a positive semidefinite matrix are always nonnegative.

(c) $i_0=i_1$ and $i_2=i_3$, i.e., the terms are of the form $a_{i_0i_0}a_{i_0i_2}a_{i_2i_2}a_{i_2i_4}a_{i_4i_0}$. The sum of such terms can be written as

$$\sum_{i_0,i_2,i_4} a_{i_0i_0}a_{i_0i_2}a_{i_2i_2}a_{i_2i_4}a_{i_4i_0} = (a_{i_0i_0})^\top_{i_0}\left(a_{i_0i_2}\sum_{i_4} a_{i_2i_4}a_{i_4i_0}\right)_{i_0i_2}(a_{i_2i_2})_{i_2}.$$

So we only need to show that the matrix contained between brackets is positive semidefinite. This follows from Schur's theorem because such a matrix is the componentwise product of $A$ with $A^2$. (Recall that any power of a positive semidefinite matrix is positive semidefinite.)

(d) $i_0 = i_2$ and $i_3 = i_4$, i.e., the terms are of the form $a^2_{i_0 i_1} a^2_{i_0 i_3} a_{i_3 i_3}$, and are nonnegative also in this case, since the diagonal entries are nonnegative.

(e) $i_0 = i_2$ and $i_1 = i_4$, i.e., the terms are of the form $a^3_{i_0 i_1} a_{i_0 i_3} a_{i_1 i_3}$. Consider the sum over $i_0, i_1$, for a fixed $i_3$

$$\sum_{i_0, i_1} a^3_{i_0 i_1} a_{i_0 i_3} a_{i_1 i_3} = (a_{0_1 i_3})^\top_{i_0} \left(a^3_{i_0 i_1}\right)_{i_0, i_1} (a_{i_1 i_3})_{i_1}.$$

The matrix $\left(a^3_{i_0 i_1}\right)_{i_0, i_1}$ is positive semidefinite, as it is a componentwise power of $A$. Hence the sum is nonnegative.

3. At most two different indices.

(a) $i_0 = i_1 = i_2 = i_3$, i.e., the terms are of the form $a^3_{i_0 i_0} a^2_{i_0 i_4}$. Using double stochasticity, we get

$$\sum_{i_0, i_4} a^3_{i_0 i_0} a^2_{i_0 i_4} = \sum_{i_0} a^3_{i_0 i_0} \le n.$$

(b) $i_0 = i_1 = i_2$ and $i_3 = i_4$, i.e., the terms are of the form $a^2_{i_0 i_0} a^2_{i_0 i_3} a_{i_3 i_3}$. Since the absolute value of elements of $A$ is not greater than 1 and the diagonal elements are nonnegative, we have

$$\sum_{i_0, i_3} a^2_{i_0 i_0} a^2_{i_0 i_3} a_{i_3 i_3} \le \sum_{i_0, i_3} a^2_{i_0 i_3} = n.$$

(c) $i_0 = i_2$ and $i_1 = i_3 = i_4$, i.e., terms of the form $a_{i_1 i_1} a^4_{i_0 i_1}$. Using the properties already mentioned we get

$$\sum_{i_0 i_1} a_{i_1 i_1} a^4_{i_0 i_1} \le \sum_{i_0 i_1} a^4_{i_0 i_1} \le \sum_{i_0 i_1} a^2_{i_0 i_1} = n.$$

4. All indices equal, i.e., $i_0 = i_1 = i_2 = i_3 = i_4$, i.e., the terms are of the form $a^5_{i_0 i_0}$. These terms are nonnegative, since diagonal entries are nonnegative. ∎

**Theorem 10.** *There exists $\varepsilon > 0$ such that every $n \times n$ positive semidefinite matrix $A$ with positive elements on the main diagonal and* $\operatorname{rank} A \le \varepsilon n$ *contains a rectangular hexagon of nonzero elements with exactly one vertex on the main diagonal, ie., $G_A$ contains a 5-cycle.*

Recall that a rectangular hexagon corresponds to a circuit of length 6 with alternating directions of edges in $G_A$. When one vertex of the hexagon is on the main diagonal, the edge is contracted to a loop, thus we get a circuit of length 5. Since $A$ and hence $G_A$ is symmetric, this is the same as to say that $G_A$ contains a pentagon.

**Proof.** Assume that $A$ satisfies the assumptions of the theorem. By Lemma 3 we can assume that $A^{(2)}$ is doubly stochastic. Suppose that $A$ does not contain a 5-cycle. We consider two cases and apply Lemmas 7 and 9.

1. $\mathrm{tr}A^4 \geq \mathrm{tr}A^3$. Then

$$\mathrm{rank}A \geq \frac{(\mathrm{tr}A^4)^5}{(\mathrm{tr}A^5)^4} \geq \frac{(\mathrm{tr}A^4)^5}{((c_1+c_2+c_3)\mathrm{tr}A^4)^4}$$
$$= \frac{\mathrm{tr}A^4}{(c_1+c_2+c_3)^4} \geq \frac{n}{(c_1+c_2+c_3)^4}.$$

2. $\mathrm{tr}A^4 < \mathrm{tr}A^3$. Then

$$(\mathrm{rank}A)^2 \geq \frac{(\mathrm{tr}A^3)^5}{(\mathrm{tr}A^5)^3} \geq \frac{(\mathrm{tr}A^3)^5}{((c_1+c_2+c_3)\mathrm{tr}A^3)^3}$$
$$= \frac{(\mathrm{tr}A^3)^2}{(c_1+c_2+c_3)^3} > \frac{(\mathrm{tr}A^4)^2}{(c_1+c_2+c_3)^3} \geq \frac{n^2}{(c_1+c_2+c_3)^3}. \quad \blacksquare$$

## 5. Open problems

The most interesting problem is whether in the last theorem we can omit the assumption of the matrix being positive semidefinite. The positive answer implies large rigidity of a matrix constructed in [2]. More generally, it would suffice to prove that for any rectangular $k$-gon, with $k$ constant, instead of the rectangular hexagon. Lemma 1 does not work, but also we do not see how to generalize the example showing that there is a matrix of rank $o(n)$ without triangles to the case of 5-cycles. Here and below we tacitly assume nonzero elements on the main diagonal.

It seems that Lemma 1 should provide rectangular $k$-gons for arbitrary constant $k$, as it does for rectangles, but we are not able to prove it even for hexagons without the additional condition of nonnegative entries in the matrix. The obstacle is that we do not know how to bound terms of the form $a^3_{i_1,i_2}a_{i_3,i_2}a_{i_3,i_4}a_{i_1,i_4}$.

The form of the inequalities in Lemma 1 also suggests that we should be able to prove the existence of all cycles for positive semidefinite matrices. But in this case again new terms appear that we are not able to bound when the length of cycles is longer than 5. A related problem is whether there are small complete subgraphs in low rank positive semidefinite matrices. Rosenfeld's results gives a positive answer for the size 3. A consequence of the result of Alon and Szegedy [1] is that for some $k$, there are positive semidefinite matrices of rank $o(n)$ whose graphs of nonzero elements do not contain the complete graph on $k$ elements. This problem, however, remains open for small values of $k>3$, in particular for $k=4$.

Another possible line of research is to try to prove that there are larger $k\times k$ submatrices of nonzero elements, than just $2\times 2$. This may lead to the solution of the problem of the rank vs. communication complexity problem. What is needed (to solve it positively) is to prove that for a 0-1 matrix of sufficiently low rank there is a large monochromatic matrix. Theorem 5 can be presented in a form showing a very rudimentary case of what is needed.

**Theorem 11** **(equivalent to Theorem 5).** *Let $A$ be an $n\times n$ matrix and* $\operatorname{rank} A<n/2$. *Then $A$ contains a $2\times 2$ matrix of nonzero elements, or a $k\times l$ matrix of zeros, with $k+l>n$.*

Here we do not assume nonzero elements on the main diagonal. The point is that by an easy application of Hall's theorem, the matrix has a support iff it does not contain a $k\times l$ matrix of zeros, with $k+l>n$.

For fields of characteristic $>0$ there are even more open problems. Essentially the only information that we have so far is the following. There are symmetric matrices of rank $o(n)$ over fields of characteristic $>2$ whose graph of nonzero elements does not contain a triangle. For characteristic 2, we have only nonsymmetric such matrices without any 3-cycles. Accidentally, any symmetric matrix $A$ over $GF_2$ with ones on the main diagonal can be presented as $BB^\top$. Such a decomposition in the field or reals characterizes positive semidefinite matrices and for those we know that a triangle must occur if the rank is small. For characteristic $>0$ we do not have any positive results, as the only tool that we have, the eigenvalues, does not work. In particular we do not know, if Theorem 5 holds in finite fields, with $1/2$ possibly replaced by an $\varepsilon>0$.

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